On the convexity of nonlinear elastic energies in the right Cauchy-Green tensor

David Yang Gao,[1]   Patrizio Neff,[2]   Ionel Roventa,[3]   Christian Thiel[4]

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We present a sufficient condition under which a weak solution of the Euler-Lagrange equations in nonlinear elasticity is already a global minimizer of the corresponding elastic energy functional. This criterion is applicable to energies $W(F) = \widehat{W}(F^T F) = \widehat{W}(C)$ which are convex with respect to the right Cauchy-Green tensor $C = F^T F$, where F denotes the gradient of deformation. Examples of such energies exhibiting a blow up for $\det F \to 0$ are given.

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1 Introduction

In a recent contribution, we have stated conditions for an elastic energy $W(F) = \widehat{W}(C)$ to be convex in terms of the right Cauchy-Green deformation tensor $C = F^T F$. These conditions include dependencies on the logarithm of the determinant, showing the necessary singular behavior for $\det F \to 0$; see [11] for more details.

This subject has been taken up by Spector [17] and Šilhavý [16], who presented alternative proofs. The main result is the following. Here and throughout, $\mathrm{Sym}_+(n)$ denotes the set of positive definite symmetric $n \times n$–matrices and $\mathbb{R}_+ := (0, \infty)$.

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[1]School of Applied Science, Federation University Australia, Mt Helen VIC 3350, Australia, email: d.gao@federation.edu.au

[2]Patrizio Neff,   Head of Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany, email: patrizio.neff@uni-due.de

[3]Department of Mathematics, University of Craiova, A. I. Cuza Street, Nr. 13, 200585 Craiova, Romania, email: ionelroventa@yahoo.com

[4]Corresponding author: Christian Thiel,   Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany, email: christian.thiel@uni-due.de

Theorem 1.1. *Let $f \in C^2(\mathbb{R}_+)$. The function $C \mapsto f(\det C)$ is convex in C on the set $\mathrm{Sym}_+(n)$ if and only if*

$$f''(s) + \frac{n-1}{n\,s}\,f'(s) \geq 0 \quad and \quad f'(s) \leq 0 \quad for \ all \ \ s \in \mathbb{R}_+\,. \tag{1}$$

In particular, these conditions are satisfied for $f(s) = -\log s$.

In this short note we would like to substantiate our claim from [11] that convexity in C should be somehow "nice". It is clear that convexity of the strain energy in C is independent of both the *rank-one convexity* condition [15] and Ball's *polyconvexity* condition [1]. Therefore, convexity of $\widehat{W}$ as function of C is not sufficient for establishing the existence of minimizers via the direct methods of the calculus of variations. Of course, in a neighborhood of the identity, an existence proof based on the implicit function theorem is always possible [3].

However, in [11] we have given an example of a strain energy $W(F) = \widehat{W}(C)$ which is both polyconvex with respect to F and convex in C; furthermore, the energy shows the correct behaviour for infinite compression, i.e. $\widehat{W}(C) \to \infty$ as $\det C \to 0$. Convexity with respect to C is also used profitably in the derivation of some models in elasto-plasticity, see [8].

2 Applications of the convexity with respect to C

We consider the problem of minimizing the elastic energy $I(\varphi) = \int_\Omega W(\nabla\varphi)\,\mathrm{d}x$, where $\varphi\colon \Omega \to \mathbb{R}^3$ is the deformation of an elastic body $\Omega \subset \mathbb{R}^3$, subject to the boundary condition $\varphi|_\Gamma = \phi_0$ with $\Gamma \subset \partial\Omega$, see [2]. The corresponding formal Euler-Lagrange equation is $0 = \mathrm{Div}\,\mathrm{D}_F W(F) = \mathrm{Div}\,S_1(F)$, where $F = \nabla\varphi$ denotes the gradient of deformation, $S_1(F) = \mathrm{D}_F W(F) = F\,S_2(F^T F)$ is the first Piola-Kirchhoff stress tensor and $S_2(C) = 2\,\mathrm{D}_C \widehat{W}(C)$ is the second Piola-Kirchhoff stress tensor. The following simple observation extracted from [5, 7] shows the applicability of our convexity condition.

Proposition 2.1. *Assume that $\widehat{W}$ is convex in C and that φ_0 is a weak solution of the corresponding Euler-Lagrange equation. If the second Piola-Kirchhoff stress tensor $S_2(C_0) = 2\,\mathrm{D}_C\widehat{W}(C_0)$ corresponding to $C_0 = \nabla\varphi_0^T \nabla\varphi_0$ is positive definite everywhere in Ω, then this solution is a global minimizer.*

Remark 2.2. Note that the condition of positive definiteness only refers to the second Piola-Kirchhoff stress *evaluated at the weak solution φ_0* of the Euler-Lagrange equation.

Convexity with respect to C is tantamount to the monotonicity of the second Piola-Kirchhoff stress tensor S_2, i.e. to the condition

$$\langle S_2(C_1) - S_2(C_2),\, C_1 - C_2 \rangle \; \geq \; 0,$$

where $\langle X, Y \rangle = \mathrm{tr}(Y^T X)$ denotes the canonical inner product on $\mathbb{R}^{n\times n}$. This does not violate any physical principle, whereas monotonicity of the first Piola-Kirchhoff stress

tensor S_1 (i.e. convexity with respect to the deformation gradient F) must be excluded to ensure physically plausible material behaviour.

The positive definiteness of the second Piola-Kirchhoff stress $S_2(C_0)$ at φ_0, on the other hand, means that all occurring forces, transformed to the reference configuration, are of a "tensional" type, i.e. that the solution φ_0 must not exhibit any "compressional" forces. Note, again, that this condition only refers to the solution φ_0 and that we do not require $S_2(C) = 2\, \mathrm{D}_C \widehat{W}(C)$ to be convex for *all* $C \in \mathrm{Sym}_+(3)$, which together with the convexity of $\widehat{W}$ in C would imply convexity of W in F.

Proof. We write φ as $\varphi = \varphi_0 + u$. Then $\nabla\varphi = \nabla\varphi_0 + \nabla u$ and thus

$$\nabla\varphi^T \nabla\varphi \;=\; (\nabla\varphi_0 + \nabla u)^T (\nabla\varphi_0 + \nabla u) \;=\; \nabla\varphi_0^T \nabla\varphi_0 + \nabla\varphi_0^T \nabla u + \nabla u^T \nabla\varphi_0 + \nabla u^T \nabla u\,.$$

We therefore find $\nabla\varphi^T \nabla\varphi - \nabla\varphi_0^T \nabla\varphi_0 \;=\; \nabla\varphi_0^T \nabla u + \nabla u^T \nabla\varphi_0 + \nabla u^T \nabla u$. The convexity of $\widehat{W}$ with respect to C implies

$$\widehat{W}(C) - \widehat{W}(C_0) \;\geq\; \langle \mathrm{D}_C \widehat{W}(C_0), C - C_0 \rangle \qquad \text{for all}\;\; C \in \mathrm{Sym}_+(n)\,. \tag{2}$$

Using (2) we obtain the following minimizing estimates:

$$\int_\Omega \widehat{W}(\nabla\varphi^T \nabla\varphi) - \widehat{W}(\nabla\varphi_0^T \nabla\varphi_0)\, \mathrm{d}x$$

$$\geq \int_\Omega \langle \mathrm{D}_C \widehat{W}(\nabla\varphi_0^T \nabla\varphi_0), \nabla\varphi_0^T \nabla u + \nabla u^T \nabla\varphi_0 + \nabla u^T \nabla u \rangle\, \mathrm{d}x$$

$$= \int_\Omega 2\langle S_1(\nabla\varphi_0), \nabla u \rangle + \frac{1}{2}\, \langle S_2(C_0), \nabla u^T \nabla u \rangle\, \mathrm{d}x$$

$$= \frac{1}{2} \int_\Omega \langle S_2(C_0), \nabla u^T \nabla u \rangle\, \mathrm{d}x \;\geq\; 0\,, \tag{3}$$

where the equality and inequality in (3) follow from the fact that $\int_\Omega 2\langle S_1(\nabla\varphi_0), \nabla u\rangle \mathrm{d}x = 0$ due to φ_0 satisfying the weak form of Euler-Lagrange equation and the assumed positive definiteness of $S_2(C_0)$, respectively. ∎

If the convexity condition in Proposition 2.1 is strengthened to the uniform convexity requirement $D_C^2 \widehat{W}(C).(H,H) \geq c^+\, \|H\|^2$ for all symmetric matrices H and some $c^+ > 0$, and if the solution φ_0 is a diffeomorphism, then it is easy to show that

$$\int_\Omega D_F^2 W(\nabla\varphi_0).(\nabla\vartheta, \nabla\vartheta)\, \mathrm{d}x \;\geq\; \widetilde{c}^+\, \|\nabla\vartheta\|_{L^2(\Omega)}^2$$

for all test functions $\vartheta \in C_0^\infty(\Omega)$ and some $\widetilde{c}^+ > 0$, implying the local stability of the solution [12].

We will apply Proposition 2.1 to a modified version of W as given in [11].

Proposition 2.3. *Let*

$$W(F) = \widehat{W}(C) = \widehat{W}(F^T F) = \alpha\big(\mathrm{tr}(C)\big)^2 + \beta\, \mathrm{tr}(C^2) - \log \det C$$

with $6\alpha+2\beta-1 = 0$ (which is strictly convex in C and polyconvex provided that $\alpha, \beta > 0$). Assume that $\widehat{W}(C)$ admits a smooth solution φ_0 of the Euler-Lagrange equation satisfying $\det \nabla \varphi_0 \geq C^+ > 0$ and $S_2(\nabla \varphi_0)$ is positive definite everywhere. Then φ_0 is a global minimizer.

The condition $6\alpha + 2\beta = 1$, $\alpha, \beta > 0$ ensures that the condition $D_C \widehat{W}(\mathbb{1}) = 0$ for a stress free reference configuration is fulfilled. Similarly, for the energy $\widehat{W}(C) = \alpha \operatorname{tr}(C) + \beta \operatorname{tr}(C^2) - \log \det C$, we need to impose the conditions $\alpha + 2\beta = 1$, $\alpha > 0$ and $\beta > 0$. Since the energy W given in Proposition 2.3 is polyconvex and coercive in the Sobolev space $W^{1,4}$ if $\alpha > 0$, it follows from the direct methods of the calculus of variations [2] that there exists a global minimizer to the boundary value problem considered in Proposition 2.1. The conditions under which a global minimizer satisfies the weak form of Euler-Lagrange equation are, however, not clear since it is not known whether $\det \nabla \varphi_0 \geq c^+$ for some $c^+ > 0$. On the other hand, most computational methods will try to solve the Euler-Lagrange equations. Thus, in general, it is not known whether a global minimizer has been found.

The energy

$$\widehat{W}(C) = \mu \Big(\alpha \big(\operatorname{tr}(C) \big)^2 + \beta \operatorname{tr}(C) - \log \det C \Big)$$

is a special member of the compressible Neo-Hooke family. It generates the linear elastic response $\mu \, \| \operatorname{dev} \varepsilon \|^2 + \mu \Big(2\, \alpha + \frac{1}{3} \Big) [\operatorname{tr}(\varepsilon)]^2$ with an induced linear Poisson coefficient $\nu = \frac{1}{2} \frac{4\alpha}{4\alpha+1} \in (0, \frac{1}{2})$. Here, $\| \, . \, \|$ denotes the Frobenius matrix norm and ε is the linearized strain tensor. Our examples show that the search for further insight into the properties of specific elastic energies might lead to unexpected new results. Certainly, the class of convex energies in C is much to narrow from an application point of view. Since, however, all invariance principles and side conditions have been respected, our examples show that there may be enough space for more discoveries.

A more general application of the convexity in C was given by Le Dret and Raoult [10]: they showed that if $W(F) = \widehat{W}(C)$ is convex with respect to C, then the *quasiconvex hull QW* of W is given by an analogue of *Pipkin's formula* [13]:

$$QW(F) = \inf_{S \in \operatorname{Sym}_+(3)} \widehat{W}(F^T F + S). \tag{4}$$

3 Further examples

A trivial example of a function $\widehat{W}$ which is convex in C is the Saint Venant-Kirchhoff energy [14, 9, 6]

$$\widehat{W}_{\mathrm{SVK}}(C) \;=\; \frac{\mu}{4} \| C - \mathbb{1} \|^2 + \frac{\lambda}{8} \operatorname{tr}(C - \mathbb{1})^2 \,,$$

which is convex in C as long as $\mu > 0$, $3\lambda + 2\mu > 0$. However, $\widehat{W}_{\mathrm{SVK}}$ is not rank-one convex, as has been shown by Raoult [14].

For another class of examples, consider an energy function satisfying the *Valanis-Landel hypothesis*, i.e. an energy of the form

$$W(F) = w(\lambda_1) + w(\lambda_2) + w(\lambda_3)$$

where $w\colon \mathbb{R}_+ \to \mathbb{R}$ is a fixed function and $\lambda_1, \lambda_2, \lambda_3$ are the singular values of F, i.e. the eigenvalues of $U = \sqrt{C}$. Since the eigenvalues of U are the square roots of the eigenvalues of C, the function W can be represented in terms of the eigenvalues of C via $W(F) = \widetilde{w}(\lambda_1^2) + \widetilde{w}(\lambda_2^2) + \widetilde{w}(\lambda_3^2)$ with $\widetilde{w}(t) = w(\sqrt{t})$. Then according to a result by Davis [4], W is convex with respect to C if and only if $\widetilde{w}$ is convex. If $f\colon \mathbb{R}_+ \to \mathbb{R}$ satisfies the requirements of Theorem 1.1, then an energy of the form

$$W(F) = \widetilde{w}(\lambda_1^2) + \widetilde{w}(\lambda_2^2) + \widetilde{w}(\lambda_3^2) \;+\; g(\det C)$$

is convex in C as well if $\widetilde{w}$ is convex.

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